Models and Modeling Perspectives on the Development of Students and Teachers

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This special issue of *Mathematical Thinking and Learning* describes models and modeling perspectives toward mathematics problem solving, learning, and teaching (Lesh & Doerr, 2003). The term “models” here refers to purposeful mathematical descriptions of situations, embedded within particular systems of practice that feature an epistemology of model fit and revision. That is, “modeling” is a process of developing representational descriptions for specific purposes in specific situations. It usually involves a series iterative testing and revision cycles in which competing interpretations are gradually sorted out or integrated or both—and in which promising trial descriptions and explanations are gradually revised, refined, or rejected. The latter emphasis on the “fitness” of models is critical because it suggests that models are inherently provisional, and it emphasizes that they are developed for specific purposes in specific situations—even though they may endure for longer periods of time, and even though they generally are intended to be sharable and reuseable in a variety of structurally similar situations.

The distinction between model and world is not merely a matter of identifying the right symbol-referent matches; rather, it depends intimately on the accumulation of experience and its symbolic representations over time. Models bootstrap
the world and the world “pushes back” toward revision of one’s models. The result is a “mangle of practice” (Pickering, 1995) in which argument, mathematics, and nature are comingled.

In this special issue, we are concerned not only with mature forms of models and modeling in communities of scientists and mathematicians, but also with the need to initiate students in these forms of thought. When attention is turned toward students, it is important to account for factors that lead students to recognize the need for a given type of model—and to account for students’ developing models and for their developing participation in practices that involve the invention and revision of models. These practices are constituted to negotiate the interchange between the world and its mathematical counterparts.

The four contributions to this special issue suggest a variety of ways that students (children through adults) can be introduced to highly productive forms of modeling practices. Collectively, the contributions illustrate how modeling activities often lead to remarkable mathematical achievements by students formerly judged to be too young or too lacking in ability for such sophisticated and powerful forms of mathematical thinking. They also illustrate how modeling activities often create productive interdisciplinary niches for mathematical thinking, learning, and problem solving that involve simulations of similar situations that occur when mathematics is useful beyond school. Petrosino, Lehrer, and Schauble lead off by describing how elementary school children (Grade 4 in the United States) developed a conceptual system about the notion of distribution as a model for structuring errors of measure. These children then considered distributions of results obtained via experiment in light of their emerging understanding of the structure of error as distributed. Lesh and Harel focus on the developing knowledge of middle school or high school students—and on their abilities to engage in activities that involve a variety of different types of proportional reasoning. Schorr and Clark shift attention toward the developing knowledge of teachers as exhibited in the ways that they make sense of their students’ work in model eliciting activities (Lesh, Hoover, Hole, Kelly, & Post, 2000). Finally, Doerr, Lesh, Carmona, and Hjalmarsen provide a brief overview of several of the most significant ways that models and modeling perspectives imply the need to reconsider a number of currently popular “constructivist” views about: (a) the nature of children’s developing mathematical knowledge; (b) the nature of real life situations (beyond school) where elementary, but powerful mathematical constructs, are useful; (c) the nature of mathematical understandings and abilities that contribute to success in the preceding problem-solving situations; and (d) the nature of teaching and learning situations that contribute to the development of the preceding understandings and abilities. In particular, models and modeling perspectives highlight the inherent reflexivity of collective and individual thought by naturally incorporating both social and individual perspectives on the nature of mathematical knowledge—much in the manner suggested by “pragmatists” such as Mead (1910) and Dewey (1982) nearly a century ago.
WHAT ARE DISTINCTIVE CHARACTERISTICS OF MODELS AND MODELING PERSPECTIVES?

In mathematics education, models and modeling perspectives emphasize the fact that “thinking mathematically” is about interpreting situations mathematically at least as much as it is about computing. Or, in the case of mathematics teacher education, models and modeling perspectives emphasize that expertise in teaching is reflected not only in what teachers can “do,” but also what they “see” in teaching, learning, and problem-solving situations. That is, it involves not only doing things right, but also doing the right things at the right time and with the right people. For example, expertise in teaching involves the development of powerful conceptual tools for making sense of students’ work. Similarly, in the case of school children, the development of elementary, but powerful mathematical models, should be considered to be among the most important goals of mathematics instruction. So, answers to the question: “What mathematical knowledge and abilities has a given student developed?” should include not only information about “What computations can they do?” but also information about “What kinds of situations can they describe mathematically?” It should include information about “What models have they developed?” as well as information about “What skills have they mastered?”

Like most contemporary theories in cognitive science, models and modeling perspectives begin with the assumption that humans interpret their experiences using internal conceptual systems (or constructs) whose functions are to select, filter, organize, and transform information, or to infer patterns and regularities beneath the surface of things (Lesh & Doerr, 2003). Yet, whereas some cognitive theories speak about these interpretive or descriptive systems as if they resided totally within the minds of students, models and modeling perspectives recognize that to have sufficient power for dealing with realistically complex problem-solving situations, relevant conceptual systems usually must be expressed using a variety of interacting media that may range from spoken language, to written symbols, to diagrams, to experience-based metaphors, to computer-based simulations (Cramer, 2003; Johnson & Lesh, 2003). In short, thinking is a form of mediated activity (Wertsch, 1985); and, because different media generally emphasize and deemphasize somewhat different aspects of the underlying conceptual systems (and the situations that they are used to describe), the meanings of these conceptual systems often are distributed across a variety of interacting media. In this sense, modeling is a form of literacy (diSessa, 1998; Gee, 1997; Olson, 1994).

FOUNDATIONS OF MODELS AND MODELING PERSPECTIVES

As Figure 1 suggests, mathematical models are conceptual systems that are: (a) expressed for some specific purpose (which John Dewey referred to as an
“end-in-view”), and (b) expressed using some (and usually several) representational media. That is, mathematical models are purposeful descriptions or explanations. They focus on patterns, regularities, and other systemic characteristics of structurally significant systems. Their purposes often involve constructing, manipulating, or predicting the systems that are being modeled; and, the process of developing sufficiently useful models for a specific purpose usually involves a series of iterative testing and revision cycles (Lesh & Doerr, 2003).

As students go through a series of modeling cycles during a given modeling activity, what we expect to observe is the emergence of a series of systematically different ways of thinking about the nature of the objects, relations, operations, and patterns or regularities in the problem-solving situation. This is because mathematical models, and their underlying conceptual systems, generally are defined by specifying four components.

- The nature of their mathematical “objects” (e.g., quantities, shapes, locations),
- The nature of their mathematical relations among “objects,”
- The nature of their mathematical operations on “objects,”
- The nature of their mathematical patterns and regularities that govern the preceding objects, relations, and operations.

According to Figure 1, mathematical models involve:

- purposes,
- underlying conceptual systems,
- media in which the conceptual system is expressed.
Because models are developed for specific purposes in specific situations, they involve situated forms of learning and problem solving (Greeno, 1991). On the other hand, the need to develop models (or other conceptual tools) seldom arises unless part of the goal includes sharability (with others) and reusability (in other situations). Therefore, modeling is inherently a social enterprise, and significant forms of generalizability and transferability are involved.

AN EXAMPLE OF A MODEL-ELICITING ACTIVITY FOR MIDDLE SCHOOL MATHEMATICS

In research on models and modeling in school classrooms, the kind of model-eliciting activities that are referred to in the preceding section usually are simulations of “real life” situations in which mathematical thinking is needed for success (Lesh et al., 2000). One characteristic of such problems is that the products that students produce are not restricted to brief answers to artificially restricted questions about premathematized situations. For example, the goal often is to develop a conceptual tool that goes beyond being useful for some specific purpose in a given situation—and that also is sharable (with others) and reuseable (in other similar situations). What’s problematic about most model-eliciting activities is that students must make symbolic descriptions of meaningful situations. Consequently, it is not surprising that model-eliciting activities tend to emphasize quite different understandings and abilities than those that are emphasized on traditional textbook word problems—where students must make sense of symbolically described situations.

The Paper Airplane Problem that follows is a middle school version of a “case study” that we first saw being used in Purdue’s graduate program for aeronautical engineering—where students used a wind tunnel to develop a useful operational definition for the concept of “drag” for various shapes of planes and wings.

Note: Here, our commitment to pedagogical design focused on the question: “How can we design a problem that will allow middle school students to experience some of what we observed with the college engineering students?”

When seventh-grade students began to work on The Paper Airplane Problem, they read a newspaper article that described how to make a variety of different types of paper airplanes. Then, they made several of these airplanes and tested the flight characteristics using three different types of flight paths (Figure 2). For each flight path, the goal was to hit a designated target, starting from a given starting point, and traveling around some obstacle (such as a chair). Then, each team of students was given a data sheet (like the one shown in Table 1) showing results that were produced by “another group of students” in another school. That is, for each paper airplane, results were shown for each of three flight paths; and, for each
flight path, the following three measurements were recorded: (1) the total distance flown, (2) the distance from the target, and (3) the time in flight. The goal of the problem was for students to write a letter to students in another class describing how such data can be used to assess paper airplanes for following four kinds of flight characteristics: (1) best floater (i.e., going slowly for a long time), (2) most accurate, (3) best boomerang, and (4) best overall.

Notice that The Paper Airplane Problem is similar to many familiar situations that occur in “real life” situations when people need to think quantitatively about things that they can’t see—or cannot measure directly. Examples include most kinds of rates (speeds, exchange rates); and, they also include indexes of things such as the “productivity” for people, products, or strategies. On our website (http://tcct.soe.purdue.edu/resources/) is a transcript showing a solution that one group of average ability seventh grade students developed for this problem. As the transcript shows, solutions to model-eliciting activities often involve sorting out and integrating concepts associated with a variety of different topic areas in mathematics and the sciences. They usually involve several modeling cycles in which trial solutions are gradually refined, revised, or rejected. They generally require students to use a great many elementary mathematical ideas and skills that are not addressed in school textbooks and tests. And, in particular, they often emphasize multimedia representational fluency, as well as a variety of mathematical abilities related to argumentation, description, and communication—as well as abilities needed to plan, monitor, and assess progress while working in teams of diverse specialists. In short, because such tasks emphasize a broader range of mathematical abilities than those emphasized tests that are easily scoreable, they often enable a broader range of students to emerge as being exceptionally capable (Lesh, 2001; Zawojewski & Bowman, 2001–2004).
TABLE 1
Paper Airplane Data Table

Which team’s plane should will the prize as being the best floater?

<table>
<thead>
<tr>
<th>Team</th>
<th>Amount of Time in Air (sec)</th>
<th>Length of Throw (m)</th>
<th>Distance From Target (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Team 1</td>
<td>3.1</td>
<td>11.0</td>
<td>1.8</td>
</tr>
<tr>
<td>Team 2</td>
<td>3.8</td>
<td>10.9</td>
<td>1.7</td>
</tr>
<tr>
<td>Team 3</td>
<td>4.2</td>
<td>12.6</td>
<td>4.5</td>
</tr>
<tr>
<td>Team 4</td>
<td>2.3</td>
<td>7.3</td>
<td>3.25</td>
</tr>
<tr>
<td>Team 5</td>
<td>4.9</td>
<td>7.9</td>
<td>2.8</td>
</tr>
<tr>
<td>Team 6</td>
<td>0.2</td>
<td>1.8</td>
<td>8.8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Amount of Time in Air (sec)</th>
<th>Length of Throw (m)</th>
<th>Distance From Target (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Team 1</td>
<td>2.5</td>
<td>7.7</td>
</tr>
<tr>
<td>Team 2</td>
<td>3.2</td>
<td>9.2</td>
</tr>
<tr>
<td>Team 3</td>
<td>4.0</td>
<td>10.8</td>
</tr>
<tr>
<td>Team 4</td>
<td>1.3</td>
<td>4.9</td>
</tr>
<tr>
<td>Team 5</td>
<td>2.7</td>
<td>10.7</td>
</tr>
<tr>
<td>Team 6</td>
<td>1.8</td>
<td>3.9</td>
</tr>
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<th>Team</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Team 1</td>
<td>0.7</td>
<td>1.8</td>
<td>6.8</td>
</tr>
<tr>
<td>Team 2</td>
<td>2.3</td>
<td>8.1</td>
<td>6.1</td>
</tr>
<tr>
<td>Team 3</td>
<td>2.4</td>
<td>10.8</td>
<td>5.5</td>
</tr>
<tr>
<td>Team 4</td>
<td>1.4</td>
<td>4.9</td>
<td>4.9</td>
</tr>
<tr>
<td>Team 5</td>
<td>2.5</td>
<td>7.7</td>
<td>5.7</td>
</tr>
<tr>
<td>Team 6</td>
<td>0.1</td>
<td>1.2</td>
<td>8.2</td>
</tr>
</tbody>
</table>

Note: The “length of throw” indicates the straight-line distance between the starting point and where the plane landed. The “target” is the finishing point where the plane should land.
As models and modeling perspectives have emerged as significant conceptual frameworks for systemic research on the interacting development of students, teachers, curriculum materials, and programs of instruction, three distinct, but closely related lines of inquiry have been emphasized (Lesh & Doerr, 2003). A brief description follows for each of the three. All three are represented in this special issue of Mathematical Thinking and Learning, and, all three are elaborated and extended in a new book that is titled Beyond Constructivism: Models and Modeling Perspectives on Mathematics Problem Solving, Learning, and Teaching (Lesh & Doerr, 2003).

Research on Problem Solving Beyond School

The first line of inquiry has focused on model-eliciting problems (Lesh et al., 2000) and on two major themes: (a) identifying the mathematical understandings and abilities that are needed for success when “mathematical thinking” is needed beyond school in a technology-based age of information, and (b) identifying students who have extraordinary abilities that may not have been apparent based on past records of low performance on the narrow and shallow band of tasks emphasized in traditional textbooks and tests (Lesh, 2001). Significant findings from this research include the facts that: (a) the mathematical understandings and abilities that are emphasized on standardized tests typically represent only a remarkably narrow and shallow subset of those needed for success beyond school in a technology-based age of information (Lesh, 2001), and (b) when assessments recognize the importance of a broader range of mathematical understandings and abilities, of the type that are needed for success beyond school in a technology-based age of information, a broader range of students naturally tend to emerge as having exceptional potential (Lesh, Zawojewski, & Carmona, 2003).

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1The book, Beyond Constructivism (Lesh & Doerr, 2003), contains chapters written by more than thirty leading researchers inside and outside the field of mathematics education. Collaborations leading to these publications grew out of three multisite research groups: (a) the Models and Modeling Working Group associated with the University of Wisconsin National Center for Improving Student Learning and Achievement in Mathematics and Science Education (NCISLA), (b) the Models and Modeling Working Group associated with the North American Group for the Psychology of Mathematics Education (PME-NA), and (c) Purdue University’s Center for Twenty-first Century Conceptual Tools (TCCT).
Research on Design Principles for Productive Modeling Activities for Learning in School

The second line of inquiry is informed by the first. It focuses on the question: “If models and modeling practices are to be introduced to schools, how can this be best accomplished?” This orientation leads to investigations of design principles for productive modeling environments in school. Consequently, this line of inquiry often focuses on: (a) designing learning environments in which students develop deeper and higher-order understandings of powerful ideas in elementary mathematics and science, and (b) investigating ways to use new conceptual technologies to help make extraordinary achievements accessible to ordinary children.

This second line of inquiry often focuses on teacher–student interactions as much as on student–student interactions (Zawojewski, Lesh, & English, 2003); and, it also often focuses on the development of whole classrooms of students as learning communities that develop productive shared norms about the nature of scientific argumentation and justification (Lehrer & Schauble, 2003; McClain, 2003; vanReeuwijk & Wijers, 2002). Significant findings from this research include the fact that surprisingly young children, as well as students from highly disadvantaged backgrounds, often produce high quality results that are far more impressive than anything that would have been predicted based on results from their prior work in traditional textbooks and tests (Carlson, Larsen, & Lesh, 2002; Doerr & Lesh, 2002; Kaput & Schorr, in press, Lehrer & Schauble, 2002; Lesh, 2001; Shternberg & Yerushalmy, 2003).

Overall, this second line of inquiry complements the first. Both are attuned to the intersection of mathematics and science—as well as to the integration of diverse topics within mathematics or science. Both are aimed at helping students develop mathematical models, or sense-making systems, as a form of explanation of the natural world. Both focus on promoting conceptual development by putting students in situations where they repeatedly express-test-and-revise their own current ways of thinking about “big ideas” in mathematics or science—rather than simply being led to adopt teachers’ prefabricated ways of thinking. And, both strive for deep treatments of a small number of “big ideas”—rather than being preoccupied with superficial “coverage” of a large number of lower-level facts and skills.

Research on the Nature of Teachers’ Developing Knowledge and Abilities

The third line of inquiry shifts attention toward the nature of teachers’ developing knowledge and abilities (Clark & Lesh, 2003; Doerr & Lesh, 2002; Lehrer & Schauble, 2000; Schorr & Lesh, 2003). Results from this research include details about significant ways that most teachers’ mathematical understandings need to be
enhanced to help their students express-test-and-revise their thinking in productive directions (Schorr & Lesh, 2002).

Based on models and modeling perspectives, such research often involves on-the-job classroom-based professional development activities in which model-eliciting activities for students provide contexts in which teachers’ teaching experiences become productive learning experiences to support teacher development (Clark Koellner & Lesh, 2003; Schorr & Lesh, 2003). Keys to the success of this approach include the following.

- Model-eliciting activities for students are activities in which students repeatedly express their current ways of thinking in forms that are visible to teachers—and to the students themselves. Therefore, as students repeatedly express, test, and revise their ways of thinking, they automatically produce auditable trails of documentation that reveal important things about the constructs and conceptual systems that they are developing. In this way, such activities are thought-revealing activities.

- Research on cognitively guided instruction has shown that one of the most effective ways to help teachers improve their teaching is to help them become familiar with their students’ evolving ways of thinking about important ideas and abilities that they want their students to develop (Carpenter, Fennema, & Romberg, 1993).

- At the same time that students are developing powerful conceptual tools to make sense of model-eliciting activities, the thought-revealing nature of their responses provides opportunities for teachers to develop sharable and reusable tools that colleagues can use to observe, document, or make sense of students’ work. … In research based on models and modeling perspectives, these teacher-level tools have included:
  
  - observation forms to gather information about the roles and processes that contribute to students’ success,
  - ways of thinking sheets to identify strengths and weaknesses of products that students produce—to help teachers provide appropriate feedback and directions for improvement,
  - quality assessment guides for assessing the quality of alternative products that students produce,
  - guidelines for conducting mock job interviews based on students’ portfolios of work produced during case studies for kids—and focusing abilities valued by employers in future-oriented professions.

Thus, thought-revealing activities for students often provide contexts for equally thought-revealing activities for teachers (or parents, policy makers, professors, professionals in business and industry).
In research conducted using models and modeling perspectives, the three lines of inquiry that have been described in this section often are not conducted in isolation. They are linked. For example, research on teacher development often takes place in the context of research on student development; or, research on problem solving (outside of school) often takes into account results from research on ideas and abilities being developed in school. So, it is important to investigate the interacting development of students, teachers, and programs (Lesh, 2002)—and interactions between learning and problem solving.

Several recent research publications describe multitier research designs that were explicitly created for use in research conducted using models and modeling perspectives (Kelly & Lesh, 2000; Lesh, 2001). In these multitier design studies, students may develop thought-revealing conceptual tools for use in mathematical problem-solving situations at the same time that teachers (or researchers) are developing thought-revealing conceptual tools to encourage (or make sense of) students’ (or teachers’) modeling activities. Yet, whereas model-eliciting activities for students are decision-making situations in which students need to use mathematical ways of thinking in their everyday lives, model-eliciting activities for teachers focus on teachers’ classroom decision-making issues, and they require teachers to integrate mathematical, psychological, historical, and pedagogical ways of thinking (Lesh, 2001). … At all three tiers of such research designs, development is encouraged because each of the interacting participants (students, teachers, researchers) repeatedly express their current ways of thinking in the form of complex artifacts that are tested and revised repeatedly. Therefore, similar design principles apply to the creation of productive knowledge development at all three levels; and, at all three levels, the development cycles that participants go through automatically produce auditable trails of documentation that reveal important information about the nature of the constructs and conceptual systems that are being developed. In other words, such design activities contribute to development while at the same time generating documentation about the nature what is being learned (Lesh & Doerr, 2002).

THEORETICAL FOUNDATIONS FOR MODELS AND MODELING PERSPECTIVES

Models and modeling perspectives build on Piaget’s structuralist views about the holistic and constructed nature of the conceptual systems that children develop to make sense of their mathematical experiences. They build on Vygotsky’s conception of thought as mediated activity (Wertsch, 1985). And, above all, they build on foundations established by American Pragmatists such as John Dewey (1982), William James (1982), and Charles Sanders Peirce (1982) who were skeptical of “grand theories” in education—and who focused on developing a “blue collar”
conceptual framework for “real life” decision making by teachers and other educators (Lesh & Doerr, 2002).

One main idea that models and modeling perspectives adopt from Piaget is his emphasis on the holistic nature of conceptual systems that underlie people’s mathematical interpretations of experiences (Beth & Piaget, 1966). Piaget emphasized that many of the most important properties of mathematical systems-as-a-whole are not derived from properties of their constituent elements. Consequently, the development of these conceptual systems must involve more than simply assembling (or constructing, or piecing together) the elements, relations, operations, and principles that they include (Lesh & Carmona, 2002).

Instead, emergent properties at higher-level systems evolve from (and are reflectively abstracted from) systems of interactions at more primitive/concrete/enactive/intuitive levels; and, these conceptual reorganizations occur mainly when models fail to fit the experiences they are intended to describe, explain, or predict. Then, when conceptual reorganizations are required, development occurs along a variety of dimensions—concrete–abstract, simple–complex, intuitive–formal, situated–decontextualized, specific–general—where the right side of these developmental continua are not necessarily characteristics of more advanced understanding.

Practical implications of these perspectives include the fact that, from a models and modeling perspective, a main challenge for teachers is to find ways to put students in situations where they must express, test, and revise their own current ways of thinking. The book, Beyond Constructivism: Models and Modeling Perspectives of Mathematics Problem Solving, Learning, and Teaching (Lesh & Doerr, 2002), gives many examples to show that, for a given “big idea” in elementary mathematics, getting students to clearly recognize the need for the underlying construct is often a large part of what it means to “understand” the construct. A large part of the meaning of the construct comes from recognizing why it is needed—and recognizing how it is related to other relevant, but logically unrelated constructs.

2 All of Piaget’s famous conservation tasks required students to make judgments about (what mathematicians refer to as) invariance properties under a variety of different systems of operations, relations, and/or transformations. Consequently, because a property that is invariant with respect to a system is not meaningful until students begin to use the relevant systems to interpret their experiences, tasks that assess a person’s understanding of these invariance properties often are powerful tools for assessing their understanding of the relevant system (Lesh & Carmona, 2002).

3 In a given situation, the model that is most powerful and useful is not necessarily the one that is most abstract, complex, detailed, formal, decontextualized, or general. In general, in the context of specific learning or problem solving situations, models (and accompanying conceptual systems) that are “best” are those that deal appropriately with trade-offs involving factors such as simplicity and complexity, or cost and quality.
Drawing from the Vygotskian tradition of thought as mediated activity (Wertsch, 1991), models and modeling perspectives also emphasize the roles of conceptual tools, such as those that are supported by language or notational systems, that influence the power of peoples’ thinking. Because of the power of such conceptual tools, they generally have strong influences on students’ mathematical thinking from both an individual endeavor and a collective enterprise. This dual relation between the collective and individual experience has a long tradition in American Pragmatism—especially in Dewey’s “instrumentalist” interpretations of pragmatist perspectives (Dewey, 1998).

Practical implications of these perspectives include the fact that, whereas Piagetians often are interpreted as being pessimistic about the possibility of significantly influencing students’ levels of development of powerful constructs and conceptual systems, one important implication of Dewey’s instrumentalist perspectives is reflected in the Jerome Bruner’s famous claim that “Any concept can be taught to any child at any time in some intellectually respectable way” (Bruner, 1963). Although Bruner’s claim clearly is an exaggeration, his central points are straightforward. First, ideas develop as they come to be expressed using increasingly powerful conceptual tools and representational media. Second, by introducing appropriate conceptual tools, surprisingly sophisticated constructs often can be made accessible to most students, including those who have been least privileged, as long as the situations are meaningful and the appropriate conceptual tools are available (Harel & Lesh, 2002, Kaput & Schorr, in press; Lehrer & Schauble, 2000).

Another main idea that models and modeling perspectives adopt from Vygotsky is the concept of zones of proximal development (Vygotsky, 1978; Wertsch, 1985). That is, ideas develop, and, at any given point in time, a student’s level of understanding can be influenced by a variety of factors such as: (a) guidance provided by an adult or peer (Cobb & Yackel, 1998), (b) conceptual tools that may be available either by luck or because of interventions from an adult (Kaput, 1994), or (c) approaches suggested (or limited by) sociocultural norms and standards that have been developed by relevant communities—such as students and teachers in classrooms (McClain, 2002; vanReeuwijk & Wijers, 2003). Consequently, the instructional challenge is to help students extend, revise, reorganize, refine, modify, or adapt constructs (or conceptual systems) that they DO have—not simply to find or create constructs that they do not have (or that are not immediately available). However, whereas Vygotsky (1978) emphasized the influences of language on thought, models and modeling perspectives also recognize that language is only one among many culturally supplied conceptual tools that influence mathematical thinking (Cobb & McClain, 2001; Dewey, 1982). Furthermore, whereas Vygotsky focused on the internalization of external functions, models and
modeling perspectives recognize that development of powerful conceptual tools occurs along a variety of dimensions (Lesh, 2002). Therefore, the notion of a zone of proximal development needs to be expanded from a 1-dimensional interval to an N-dimensional region in which a variety of paths lead to any given construct. Furthermore, students are able to make progress through these regions along a variety of possible trajectories.

Practical implications of these perspectives include the fact that, particularly in the case of especially powerful constructs and conceptual systems, placing children in situations where they express, test, and revise their own ways of thinking is often quite different than leading them along narrow trajectories that lead to the (often superficial) adoption of our ways of thinking—especially if it is assumed that all children should develop along the same trajectory.

Another way that models and modeling perspectives extend Vygotsky’s ideas about the influence of social functions on conceptual development is related to Marvin Minsky’s notion of communities of mind (Minsky, 1987)—or William James’ concept of a pluriverse of conceptual systems. According to Minsky and James, in nontrivial learning or problem-solving situations, students generally should be expected to have available a community of conceptual systems—each of which have the potential to be engaged to interpret relevant experiences (Zawojewski, Lesh, & English, 2003). Thus, a student’s developing community of constructs is similar to a community of living, adapting, and continually evolving biological systems (Lesh & Doerr, 1998). Consequently, development is not likely to be encouraged by discouraging diversity; and, when the products that students produce include descriptions and explanations, then there always exist a variety of different types of responses—where trade-offs often must be considered among factors such as: precision versus accuracy, complexity versus timeliness, simplicity versus superficiality, power versus economy, or costs versus benefits.

Practical implications of these perspectives include the fact that, in a community of students, as well as in a given students’ community of potentially relevant constructs in a given problem-solving situation, diversity is to be encouraged—as long as it also is accompanied by selection, communication (so that innovations will spread), and conservation (so that innovations will be preserved). For example, in the transcripts that are given in digital appendixes to this special issue of Mathematical Thinking and Learning, it is clear that, when students make progress, they often do so by sorting out and integrating diverse (and often logically unrelated) ways of thinking. Furthermore, decisions about rejected ways of thinking often contribute as much to learning or problem solving as decisions about ways of thinking to adopt, or
refine, or revise. Consequently, a great deal of attention often is needed to ensure that ways of thinking are accepted or rejected because they are most appropriate (i.e., most useful in the given situation)—not simply because an “authority figure” says so.

CONCLUDING REMARKS ABOUT PRACTICAL ISSUES IN INSTRUCTION

Even though models and modeling perspectives can be seen as evolving naturally out of currently popular “constructivists” ways of thinking about the nature of mathematics, problem solving, learning, and teaching, they often require the development of new ways of thinking that are quite different than those traditionally adopted in mathematics education research (Lester & Kehle, 2003).

Concerning Relations Between Modeling and Problem Solving

Whereas mathematics education researchers traditionally have defined problem solving as a process of “getting from givens to goals when the solution processes are not readily available” (Schoenfeld, 1982), models and modeling perspectives emphasize that the development of models generally occurs through a series of develop-test-revise cycles—each of which involve somewhat different ways of thinking about the nature of givens, goals, and possible solution steps (Lesh & Harel, 2003). Consequently, when solution processes involve a series of modeling cycles, when early interpretations of given and goals can be expected to be naïve, and when the goal is to extend, revise, reorganize, refine, modify, or adapt constructs (or conceptual systems) that you do have rather than to function better when none are available, quite different kinds of problem-solving strategies and metacognitive processes emerge as important (Lesh, Lester, & Hjalmarson, 2003; Middleton, Lesh, & Heger, 2003; Zawojewski & Lesh, 2003).

Concerning Relations Between Applied Mathematics and Pure Mathematics

John Dewey, in particular, stressed the notion that the goal of making practice more intelligent is quite different than the goal of making intelligence more practical (Dewey, 1982). Similarly, mathematizing reality is quite different than “realizing” mathematics. This is why, models and modeling perspectives stress the importance of going beyond “teaching mathematics so as to be useful” (Freudenthal, 1973) to also help children learn to quantify, dimensionalize, coordinatize, and in other ways mathematize their experiences. On the other hand, models and modeling per-
spectives should not be interpreted as deemphasizing “pure” mathematical activities. Nor is it being suggested that, by taking students’ immature ways of thinking seriously (so that they can be explicitly tested and revised or rejected), this implies that “anything goes” and that immature ways of thinking should be celebrated while the wisdom of sages should be ignored. The whole point of emphasizing models and modeling activities is to focus on deep treatments of a small number of “big ideas” and to increase the likelihood that students will develop powerful constructs and conceptual systems—while ensuring that they will go beyond thinking with these systems to also think about them. Indeed, in a number of recent publications, we have been explicit about a variety of ways to achieve an appropriate balance between “pure” and “applied” mathematical activities (Kelly & Lesh, 2003); and, a hallmark of recent research on models and modeling is its optimism about the possibility of helping average ability students exhibit extraordinary achievements involving deeper and higher-order mathematical thinking (Lesh & Doerr, 2002). However, when we recognize that there is a need to develop mathematics out of situations that are meaningful to students, we recognize that, for “pure” mathematicians, “pure” mathematical systems are meaningful—and that many can be made so for youngsters in school. On the other hand, for children who have not yet developed most of these systems to a level that they can be explored meaningfully, there is absolutely no danger that mathematics textbooks are likely to include too many activities that could be called “model-eliciting” (Lesh et al., 2000). In fact, in classes that we teach for experienced teachers, when we give participants the assignment to review their own textbooks to try to find instances of problem-solving activities that satisfy the six design principles that define what it means to be a model-eliciting activity (Lesh et al., 2000), they typically discover that no such problems are included. That is, every problem violates every one of the six principles! Of course, model-eliciting activities are not the only kinds of problems that may be beneficial to include in instruction. Nonetheless, since model-eliciting activities were designed explicitly to focus on mathematical ideas and abilities that are needed for success beyond school in a technology-based age of information, it is noteworthy that current textbooks and tests typically include no such activities.

Concerning Relations Between Basic Skills and More Powerful Constructs

Models and modeling perspectives should not be interpreted as advocating the neglect of basic facts and skills. This is true for the same reasons that coaches need not neglect fundamentals and skills simply because their teams are allowed to scrimmage occasionally. In other publications, we have dealt extensively with this issue of attaining a balance between skill development and the development of deeper and higher-order abilities (Dark, 2003; Lesh, 2001). For now, we simply
observe that, if a teacher’s goal is to help students develop more powerful ways of thinking about that are known to be difficult to understand—such as ideas related to density, rates of growth, forces, sampling, and chance—then more is needed than simply introducing a few new facts and skills. All of these ideas presuppose the development of some specialized conceptual system; and, unless students are challenged to express their underlying ways of thinking about these constructs in forms that create the need for testing and revision, new facts and skills tend to be grafted onto inappropriate models for making sense of experiences. Our research suggests that activities that challenge students to express their ways of thinking in the form of complex artifacts provide some of the most effective instructional “leverage points” for developing powerful conceptual tools that radically amplify their mathematical abilities (Kaput & Schorr, in press; Lehrer & Schauble, 2002; Lesh & Kelly, 2002).

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