Chapter 2

Model Development Sequences

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This chapter describes instructional modules that are based on a models and modeling perspective, and that were designed to meet goals that are unusual compared with those driving the development of most commercially produced materials for instruction or assessment. First, the modules were designed to provide rich research sites for investigating the interacting development of students and teachers. Therefore, they are modularized so that components can be easily deleted, extended, modified, or resequenced to suit the needs of researchers (or teachers) representing a variety of theoretical perspectives, purposes, and student populations. Second, to make it possible to observe processes that influence the development of students’ and teachers’ ways of thinking, the modules were designed to be thought revealing (Lesh, Hoover, Hole, Kelly, & Post, 2000) and to be efficient for producing maximum results using minimum investments of time and other resources. Consequently, from the perspective of teachers, they have the unusual characteristic of seeming to be small-but-easy-to-extend rather than being large-and-difficult-to-reduce. Third, they were designed to emphasize important understandings and abilities that are needed for success beyond schools in a technology-based age of information.

Even though many of the big ideas that are especially powerful in everyday situations have long traditions of being treated as foundation-level ideas in elementary mathematics, it will be clear, in transcripts that are given throughout this book, that many others are not. Also, the activities that we’ll be describing often give special attention to levels and types of understanding (and ability) that seldom have been emphasized in traditional textbooks, tests, or teaching. Consequently, by emphasizing an unusually broad range of understandings and abilities, a broader range of students typically emerge as being highly capable; and, many of these students are surprisingly young or were formerly labeled
“below average” in ability or achievement. In short, a goal of these modules has been to help researchers investigate the nature of situations in which surprising students produce surprisingly sophisticated results at surprisingly young ages, in surprisingly brief periods of time, and with surprisingly little direct instruction.

Our experience suggests that one of the most effective ways to achieve the preceding goals is to put students in situations where their everyday knowledge and experience enables them to clearly recognize the need for the constructs that they are being challenged to produce – and where a lack of facility with esoteric facts and skills does not prevent them from using their knowledge and experience to develop the required conceptual tools.

**RELEVANT ASSUMPTIONS ABOUT THE NATURE OF MATHEMATICS**

For the *model development sequences* that are described in this chapter, an underlying assumption is that one of the most important characteristics that distinguish mathematical knowledge from other categories of constructs is that mathematics is the study of (pure)\(^1\) operational/relational structures. Therefore, we focus on the underlying structural aspects of mathematical constructs; and, we assume that a large part of the meanings for objects (such as equations, number systems, formulas, and graphs) derive from their existence as parts of mathematical systems in which they function. From this point of view:

- Doing "pure" mathematics means investigating systems for their own sake—by constructing and transforming and exploring them in structurally interesting ways, and by studying their structural properties (Steen, 1987, 1988).\(^2\)
- Doing applied mathematics means using the preceding systems as models (or conceptual tools) to construct, describe, explain, predict, manipulate, or control other systems. (Lesh & Doerr, 1998).

Jean Piaget (Piaget & Beth, 1966) was one of the most influential people to emphasize the holistic structural character of children’s mathematical reasoning; and, Zoltan Dienes (1960) was one of the most creative mathematics educators to specify principles for designing instructional activities to help children develop these structure-based understandings and abilities. Dienes focused on concrete-to-abstract dimensions of conceptual development by emphasizing

\(^1\) The word "pure" is enclosed in parentheses because these conceptual systems are never completely pure. To be useful beyond trivial situations, they are always expressed using (several) representational systems, and meanings are influenced by unstated experiences, assumptions, and purposes of the particular humans who use them.

\(^2\) Musicians, for example, begin to step into the realm of mathematics when they go beyond playing sequences of individual notes toward investigating structural properties associated with whole patterns of notes.
activities with concrete manipulatable materials such as bundling sticks, a counting frame abacus, or arithmetic blocks (sometimes called Dienes Blocks).

![Arithmetic Blocks](image)

**FIG. 2.1.** Arithmetic blocks can be used for teaching place value arithmetic.

In previous publications, we have described productive ways that Dienes’ principles can be extended to activities that involve computer-based graphics and animations rather than concrete materials (Lesh, Post, & Behr, 1987a). We also have described ways to apply Dienes’ principles to teacher education programs whose aims are to teach teachers using techniques that we want them to use to teach their children (Bell, Fuson, & Lesh, 1976). This chapter briefly describes how Dienes’ principles can be extended to include: (a) model development sequences that emphasize activities involving simulations of real life problem solving situations (that are familiar and meaningful to the students) rather than artificial mathematics laboratory materials (that may be quite abstract in spite of the concrete material), and (b) problem solving experiences that emphasize social dimensions of mathematical understanding—where students' interactions with peers or teachers are as significant interactions with concrete materials.

**A BRIEF REVIEW OF DIENES’ INSTRUCTIONAL PRINCIPLES**

Dienes used the term *embodiment* to refer to concrete manipulatable materials (such as arithmetic blocks) that are useful props to help children develop elementary-but-powerful constructs that provide powerful foundations for elementary mathematical reasoning. He also emphasized the following four principles about how embodiments should be used in instruction:

1. *The Construction Principle*: Dienes, like Piaget, believed that many of the most important constructs in elementary mathematics must be abstracted, not from concrete objects, nor from isolated actions performed on concrete objects, but from *systems* of operations and relations that must be read into a given
embodiment before the system itself can be read out. According to this point of view, mental and physical actions only become elevated to the status of mathematical operations (or relations) when reflective abstraction treats them as being part of an operational-relational-organizational system-as-a-whole. Similarly, concrete materials only become embodiments of a given mathematical system after a child has coordinated the relevant actions to function as a system-as-a-whole in the context of these materials. Thus, concrete materials serve as supports for the student's conceptual activities; but, the abstraction is from the system of conceptual actions - not from the materials in which they operate.

2. The Multiple Embodiment Principle: To help a child go beyond thinking with a given construct (or conceptual system) to also think about it, several structurally similar embodiments are needed. Also, students need to focus on similarities and differences as the relevant system-as-a-whole functions in different contexts. Thus, students must go beyond investigating individual embodiments to investigate structure-related relationships among several alternative embodiments—perhaps by making translations or predictions from one embodiment to another.

3. The Dynamic Principle: In mathematics, components and characteristics of the relevant systems often refer to dynamic operations or transformations, rather than to static objects or states. Also, many of the most important characteristics involve invariance properties, or properties such as transitivity that apply to patterns or regularities rather than to isolated objects or actions. In particular, some of the most significant objects are variables (or operators, or transformations) rather than being objects in the usual sense of the term. Therefore, it is important for the relevant systems to be viewed as being dynamic rather than static, and it is important for attention to focus on patterns and regularities rather than on isolated pieces of information.

4. The Perceptual Variability Principle: Every embodiment of a mathematical system has some characteristics that the abstract system does not; and every mathematical system has some characteristics that the embodiment does not have. Therefore, when multiple embodiments are emphasized, it is important to use materials that have different perceptual characteristics. To select a small number of especially appropriate embodiments to focus on a given construct, irrelevant characteristics should vary from one embodiment to another so that these characteristics are "washed out" of the resulting abstraction. Collectively, the embodiments that are chosen should illustrate all of the most important structural characteristics of the modeled system.

In summary, Dienes's principles were designed to help students: (a) go beyond focusing on concrete materials, or on isolated actions, to focus on patterns and regularities that occur within systems of operations and relations that are imposed on the materials, (b) go beyond focusing on isolated embodiments to focus on similarities and differences among structurally similar systems that
they come to embody, (c) go beyond static patterns and objects to focus on dynamic systems of operations, relations, and transformations, and (d) go beyond thinking with a given model to also think about it for a variety of problem solving functions. For example, as Fig. 2.2 suggests, if Dienes’ principles are used with elementary school children to focus on constructs that underlie our base-ten numeration system, a given child might investigate structural similarities among activities involving three different sets of concrete materials—such as arithmetic blocks, bundling sticks (popsicle sticks), and a counting frame abacus—and the relevant conceptual system is the system of operations and relations that is common to all three embodiments.

FIG. 2.2. Dienes’ Multiple Embodiment Principle.

**BEYOND THE PRIMARY SCHOOL, WHY DO SO FEW TEACHERS USE ACTIVITIES WITH CONCRETE MANIPULATABLE MATERIALS?**

One answer to this question results from the commonly held belief that older children will be offended if “baby things” are used in instruction. But, activities with concrete materials often involve conceptual systems that are not at all simple, concrete, intuitive, or simple. In fact, in research with teachers and undergraduate students (Fuson, & Lesh, 1976; Carlson, 1998; Doerr & Tinto, 2000; Lesh, Post, & Behr, 1987b; Bell), deep conceptual weakness often become apparent if these adults are challenged to go beyond simply using memorized rules about the manipulation of written symbols—and are challenged to explain why these rules work using diagrams, concrete materials, experience-based metaphors, or other potential embodiments. Consequently, for teachers who experience such difficulties, fears often exist that, if these materials are used in instruction, the teacher’s own misunderstandings would be exposed, and children would become confused in ways that the teachers could not explain. Beyond the preceding concerns, even when concrete manipulatable
materials have been used in instruction, teachers and textbook authors generally use Dienes’ materials without using his theory. For example:

- If concrete materials are used, they tend to be used to give demonstrations (or explanations) in which no construction activities are involved from students. Thus, both the construction principle and the dynamic principle are violated, and the relevant abstractions are expected to come from the materials themselves rather than from the (mental) actions that are performed on the materials.

- Because construction activities are omitted that would elevate the concrete materials to the status of being embodiments of the relevant conceptual system(s), the concrete materials often are used in ways that are no less abstract than using written symbolic symbol systems.

- Usually, only a single embodiment is used, and no attempt is made to explore structural similarities among related embodiments. Thus, the multiple embodiment principle is violated, and issues related to the perceptual variability principle never arise.

Because one of our goals is to extend Dienes principles to make them more useful to teachers beyond primary school, we have explored the possibility of replacing activities involving concrete materials with activities in which the contexts are grounded in the everyday experiences of students or their families. The result is depicted in Fig. 2.2 below.

![Model development sequences.](image)

Whereas Dienes’ multiple embodiment principle recommends using a series of three or more embodiments for a given construct (see Fig. 2.2), we often replace the first and the third embodiments with experiences that we call model-eliciting activities. The result, illustrated in Fig. 2.3, is a version of Dienes’ multiple embodiment principle that focuses on applied mathematics as much as pure mathematics—and on stories, diagrams, and experience-based metaphors as much as concrete materials.
MODEL DEVELOPMENT SEQUENCES INCLUDE SITUATIONS GROUNDED IN STUDENTS’ EVERYDAY EXPERIENCES

As Fig. 2.3 suggests, the first activity in model development sequences is a model-eliciting activity. Details about model-eliciting activities have been described in several past publications (e.g., Lesh et. al, 2000; Lesh, Hoover, & Kelly, 1993). In general, they are similar to the case studies that are used for both instruction and assessment in future-oriented graduate programs and professional schools—in fields ranging from aeronautical engineering, to business management, to agricultural sciences—at universities such as Purdue, Syracuse, Minnesota, or Wisconsin. In these graduate programs and professional schools, many of the most important goals of instruction involve helping future leaders develop powerful models and conceptual tools for making (and making sense of) complex systems. That is, they are simulations of real life problem solving situations; they require more than a few minutes to complete; teams of specialists often work together using powerful conceptual technologies; and, the central goal is to develop, test, revise and refine powerful, sharable, and re-useable conceptual tools that involve much more than simple answers to questions of the type emphasized in traditional textbooks and tests.

The Volleyball Problem that is given in appendix A gives an example of a model-eliciting activity that is based on a case study that we observed used in Northwestern University’s Kellogg School of Management. The original problem was designed to encourage graduate students to develop powerful ways of thinking about situations in which information about products (or places, or people) must be assigned “weights” and then aggregated in some way in order to assign “quality ratings” for some purpose. For example, in everyday situations, such weighting schemes are used (implicitly or explicitly) when automobiles are rated in a consumer guide book, cities are rated in a places rated almanac, sports teams are rated in newspapers, or students are rated based on scores on laboratory projects, weekly quizzes, unit tests, teachers’ observations of classroom participation, plus performances on midterm and final examinations.

The Volleyball Problem gives information taken from tryout activities at a sports camp that specializes in volleyball instruction for young girls. The newspaper article that sets the context for this problem describes a situation where, on the first day of the camp, the following information was gathered for each girl who wanted to participate: height, vertical jumping ability (data from three jumps), serving ability (data from five serves), speed in running a 40-yard dash (results from three races), and comments from school coaches. The goal of the problem is to write a two-page letter to the camp director describing a scheme for using information from the tryouts to rank the girls—and to assign them to teams that are equivalent in overall ability.

The Volleyball Problem elicits a model in the sense that the product that the students produce is not simply a short numeric answer to a situation that has
already been completely mathematized by others. Instead, the product is a
description that expresses students’ ways of thinking about issues such as: (a)
how to quantify relevant qualitative information (e.g., coaches comments,
weights that assign values to the relative importance of various factors), and (b)
how to aggregate information in order to operationally define an “index of
quality” for individual players and teams.

Appendix B gives a transcript that shows a typical solution process used by
middle school students who have worked on the Volleyball Problem. The
solution is typical in the following ways.

- The product that is produced goes through a series of iterative modeling
cycles in which trial ways of thinking are tested and revised repeatedly.
- The mathematical understandings and abilities that are needed for
success are quite different than those emphasized on standardized tests.
- Children who have been classified as below average (based on
performance in situations involving traditional tests, textbooks, and
teaching) often invent (or significantly revise or extend) constructs and
ways of thinking that are far more sophisticated than anybody ever dared
to try to teach to them (Lesh & Doerr, 2000). For example, in the
Volleyball Problem, average ability middle school students often invent
“weighted averages” (or “weighted sums”)—in spite of the fact that their
past performances on standardized tests suggested that they knew
nothing about simple averages (or mean values).

In short, when middle school children work on well-designed model-
eliciting activities, what we observe coincides with what professors express as
“common knowledge” about graduate students (or on-the-job adults) who work
on complex projects designed to help prepare them for leadership in future-
oriented professions in business, engineering, or the sciences. That is:

- Relevant stages of problem solving include: problem posing, information
gathering, mathematizing, planning, communicating, monitoring, and
assessing intermediate results. Therefore, the levels and types of under-
standings and abilities are emphasized include many that involve many
that go beyond traditional conceptions of content-related expertise.
- Basics from an industrial age are not sufficient to provide adequate
foundations for future success in a technology-based age of information3;
and, past conceptions of mathematics, science, reading, writing, and
communication are far too narrow, shallow, and restricted to be used as a

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3 What is advocated here is not to abandon fundamentals. This would be as foolish in mathematics
and science (or reading, writing, and communicating) as it is in basketball, cooking, or carpentry;
and, greater accountability is what we support, not what we oppose. But, it is not necessary to master
the names and skills associated with every item at Sears before students can begin to cook or to build
things; and, great basketball teams are not likely to evolve by never allowing children to scrimmage
until they had completed twelve years consisting of nothing but drills on skills. What is needed is a
sensible mix of complexity and fundamentals; both must evolve in parallel; and, one does not come
before (or without) the other.
basis for identifying students whose mathematical abilities should be recognized and encouraged.

- Students who are especially productive and capable in the preceding situations often are not those with records of high scores on standardized tests—or even high performance in traditional schooling.

To create simulations of real life problems that are especially designed to recognize and reward a broader range of students with exceptional potential in mathematics, hundreds of expert teachers have worked with us to develop the following six principles of instructional design. These principles were developed during a series of 10-week research studies investigating the development of teachers’ assumptions about: (a) the nature of real life learning and problem-solving situations in which mathematical reasoning is useful in an age of information, and (b) the nature of the understandings and abilities that are needed for success in the preceding kinds of situations (Lesh et al., 2000).

1. **The Personal Meaningfulness Principle** (sometimes called the “Reality” Principle): Could this really happen in real life situations? Will students be encouraged to make sense of the situation based on extensions of their own personal knowledge and experiences? Will students’ ideas be taken seriously, or will they be forced to conform to the teacher’s (or author’s) notion of the (only) correct way to think about the problem situation?

2. **The Model Construction Principle**: Does the task ensure that students clearly recognize the need for a model to be constructed, modified, extended, or refined? Does the task involve constructing, describing, explaining, manipulating, predicting, or controlling a structurally significant system? Is attention focused on underlying patterns and regularities rather than on surface-level information?

3. **The Self-Evaluation Principle**: Are the criteria clear to students for assessing the usefulness of alternative responses? Will students be able to judge for themselves when their responses are good enough? For what purposes are the results needed? By whom? When?

4. **The Model-Externalization Principle** (sometimes called the Model-Documentation Principle): Will the response require students to explicitly reveal how they are thinking about the situation (givens, goals, possible solution paths)? What kind of systems (mathematical objects, relations, operations, patterns, regularities) are they thinking about?

5. **The Simple Prototype Principle**: Is the situation as simple as possible, while still creating the need for a significant model? Will the solution provide a useful prototype for interpreting a variety of other structurally similar situations? Will the experience provide a story that will have explanatory power—or power for making sense of other structurally similar situations?
6. The Model Generalization Principle: Does the conceptual tool that is constructed apply to only a particular situation, or can it be modified and extended easily to apply to a broader range of situations? Students should be challenged to go beyond producing single-purpose ways to thinking to produce reusable, sharable, modifiable models.

These six principles are described in greater detail in Lesh, Hole, Hoover, Kelly, & Post (2000). For the purposes of this chapter, it is important to emphasize that, even though these principles appear to be sensible and straightforward to use, the teachers who developed them also used them to assess (or improve) activities that they found in textbooks, tests, and programs for performance assessment; and, what these teachers discovered was that, even when they started by selecting activities that seemed to be promising, nearly every task violated nearly every one of their principles—and these shortcomings were far from easy fix (Lesh et al., 2000).

Whereas activities with concrete materials typically rely on carefully sequenced “guided questioning techniques” designed to lead students to think the way the author wants them to think, model-eliciting activities encourage students to repeatedly express, test, and refine or revise their own ways of thinking. Whereas activities with concrete materials typically assume that students know almost nothing about the construct the author wants them to develop, model-eliciting activities are more closely aligned with Jerome Bruner’s famous claim that “the foundations of any subject can be taught to anybody, at any age, in some form” (Bruner, 1960, p. 12)—because, long before most constructs are understood as formal abstractions, they are used intuitively in meaningful situations. Thus, to encourage the development of such constructs, the first trick is to put students in familiar situations in which they clearly understand the need for the desired construct; the second is to ensure that responses can be based on extensions of students’ everyday knowledge and experiences; and the third is to provide meaningful “design specs” involving constraints that enable students to “weed out” inadequate ways of thinking.

MODEL DEVELOPMENT SEQUENCES
GO BEYOND RELYING ON ISOLATED PROBLEM-SOLVING ACTIVITIES

Isolated problem-solving activities are seldom enough to produce the kinds of results we seek. Sequences of structurally related activities are needed, and discussions and explorations are needed to focus on structural similarities among related activities.

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4 Novices often try to fix flawed problems by adding the phrase “explain your answer”—or by using scoring rubrics that focus on processes and underlying ways of thinking. But, such tactics are seldom effective for reasons described in detail in Lesh, Hoover, Hole, Kelly, & Post (2000).
For many of the same reasons why the Dienes’ multiple embodiment principle is needed to get the most instructional value out of embodiments of mathematical constructs and conceptual systems, model development sequences are needed to get the most instructional value out of model-eliciting activities. Also, because model-eliciting activities tend to rely much less heavily on narrowly guided questioning than most activities with concrete materials, discussions about sequences of several related activities are needed to ensure that the sequence-as-a-whole will have a clear sense of direction—even though the individual activities are relatively open-ended.

**FIG. 2.4.** A standard organizational scheme for model development sequences.

In *model-eliciting activities*, authors determine what kind of quantities and quantitative relationships (and operations) students must take into account, and they also determine which design specs provide constraints to weed out inadequate ways of thinking. Consequently, in order to get students to develop the desired conceptual tools, authors do not need to rely exclusively on guided questioning techniques. Furthermore, if discussions and explorations encourage students to investigate similarities and differences among structurally related tasks, then this provides another powerful tool to focus attention on the conceptual system that is common to all of the activities. Thus, model development sequences involve the following standard organizational scheme.

*Warm-up activities* usually are given the day before students are expected to begin work on the model-eliciting activity. Often, they are based on a math-rich newspaper article, or on a math-rich web site, which is followed by a half dozen questions aimed at:
• helping students read with a mathematical eye while also familiarizing them with the context of the model-eliciting activity—so that solutions are based on extensions of students real life knowledge and experiences, and so that time is saved that otherwise is wasted on “getting acclimated” during the model-eliciting activity.

• answering teachers’ questions about “minimum prerequisites” for students to begin working on the model-eliciting activity.

• informing parents and other interested individuals about the practical importance of conceptual tools students are developing.

Model-eliciting activities usually require at least one or two full class periods to complete, and students usually are encouraged to work in teams with three students in each group. Often teachers use model-eliciting activities at the beginning of a unit that deals with the same big idea that underlies the main construct emphasized in the model-eliciting activity. Because model-eliciting activities require students to express their ways of thinking in forms that are visible to teachers, a goal is to identify students’ conceptual strengths and weaknesses—to inform instructional decisions in much the same way as if the teacher had time to interview every student before teaching the unit.

Model-exploration activities are similar to Dienes’ embodiment activities. However, rather than relying exclusively on concrete materials, model-exploration activities often involve computer graphics, diagrams, or animations (Lesh, Post, & Behr, 1987a). But, regardless what kind of “embodiments” are used, the goal is for students to develop a powerful representation system (and language) for making sense of the targeted conceptual system. For example, for the model-exploration activity that goes with the Volleyball Problem in appendix A, the embodiment emphasizes the following three important processes.

• mapping scores for running, or jumping, or serving onto number lines.

Example: If three people have average vertical jumps of 15’, 17’, and 26’, they might be mapped to the numbers 3, 2, and 1 respectively—because they ranked 1st, 2nd, and 3rd in this event.

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Jumping Scores: |---------------|---------------|---------------|
0’   2’   4’   6’   8’   10’  12’  14’  16’  18’  20’  22’  24’  26’
15’  17’  26’

Number Line: 1st 2nd 3rd 4th 5th 6th 7th 8th 9th 10th
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• assigning weights to different quantities by stretching or shrinking the related number lines.
• investigating sums by combining lengths on several number lines.

Example: Suppose that Jen ranked 5th in jumping and 8th in running, whereas Trish ranked 8th in jumping and 5th in running; and, also suppose that the coaches consider jumping ability to be two times more important than jumping ability. Then, the arrows below show how Jen compares with Trish in overall ability—if only running and jumping abilities are considered.

Jumping Ability:

Jen:

Trish:

Running Ability:

Jen:

Trish:

Combined Running and Jumping Ability:

Jen:

Trish:

Note: According to the preceding way of measuring and combining scores, Jen is predicted to be a slightly better volleyball player than Trish because short arrows correspond to high rankings.

For the model-exploration activity that is given in appendix A, other ways also are shown for mapping running, jumping, and serving scores to number lines; and, these same number lines also can be used to score qualitative comments from coaches. For example:

• Jumping scores can be mapped to a number line that goes from zero to ten, or they can be mapped to a number line that goes from zero to one hundred. Furthermore, in either case, the scores can be rounded off and restricted to whole numbers—or they can be assigned decimal values or fractional values.
• For either scores or rankings, weights can be assigned to different quantities by multiplying the numbers, or they can be assigned by
stretching or shrinking the number lines. In any case, good scores may correspond to either high numbers or low numbers; and, regardless what kind of mapping is used, consideration must be given to the fact that short time intervals are good for running, whereas long length intervals are good for jumping.

Using any of the preceding mappings, it is possible (metaphorically speaking) to add apples (jumping distances) and oranges (running times). That is, two fundamentally different kinds of quantities can be mapped to number lines that have homogeneous units which may be interpreted as ranks (ordinal numbers) or as point scores (cardinal numbers) or as directed distances (vectors). In any case, the mapping and weighting procedures that are used tend to distort the original information in different ways that often change the overall quality rating associated with combined scores. In particular, different volleyball players may appear to be best overall depending on the specific mapping and weighting procedures that are used. Should the difference between running scores of 4.0 seconds and 5.0 seconds be treated as being the same as the difference between running scores of 5.0 seconds and 6.0 seconds? Should a 15” jump be treated as half as good as a 30” jump? There are no value free answers to such questions. Every system of measuring makes some assumptions about the underlying phenomena being measured; and, this includes raw data that records running speeds using time intervals—and that records jumping ability using distance intervals.

For the model-exploration activity that is given in appendix A, one of the big ideas is for students to become more aware of assumptions that are made any time real phenomena are mapped to number line models—and, in particular, any time weights (or lengths) are assigned to the underlying quantities, and any time scores are combined by adding, subtracting, multiplying, or dividing.

Appendix C gives a sample of different kinds of responses that middle school students often generate for the Volleyball Problem. Sometimes they may begin by finding a way to combine the scores for each volleyball player (perhaps using some sort of weighted sum, or weighted average), and then, they may rank the volleyball players using these combined scores. Or, they may begin by ranking volleyball players within each category of information; and then, they may devise a way to combine these rankings. In any case, mappings to number lines provide productive ways to think about (and visualize, and talk about) the procedures that are used—and the assumptions that are being made. This is why mappings to (and operations with) number lines were chosen to embody the conceptual systems that we wanted to highlight in the model development sequence related to the Volleyball Problem.

Model-exploration activities are intended to help students develop powerful language and representation systems that are useful to go beyond thinking with the relevant construct (or conceptual system) to also think about it. In particular, students often develop powerful conceptual tools that can be used to crush the problem they were given in the follow-up model-adaptation activity—which is the third major activity in model-development sequences.
Model-adaptation activities sometimes have been called model-application activities or model-extension activity. This is because the goal of the activity often focuses on using the conceptual tool that was developed in the model-eliciting activity (and refined in the model-exploration) to deal with a problem that probably would have been too difficult to handle before the tool was developed. However, even though the tool was developed in a form that was intended to be easily sharable, and modifiable, and transportable, it seldom can be used in a new situation without making some significant adaptations. Furthermore, even though many model-adaptation problems are essentially more difficult versions of model-eliciting activities, many focus on problem posing at least as much as on problem solving—and on information gathering at least as much as on information processing.

Teachers often think of both model-eliciting activities and model-adaptation activities as being performance assessment activities—because both are similar to real life situations in which mathematics is used outside of school, both contribute to both learning and assessment, and both emphasize deeper and higher-order understandings and abilities that are seldom addressed on standardized tests. Consequently, teachers often think of model-eliciting activities often as pretests, and model-adaptation activities as post-tests for a given unit—where both focus on the main big ideas emphasized in the unit.

Another difference between model-eliciting activities and model-adaptation activities is that, for the former, students typically work in teams consisting of three to four people; whereas, for the latter, students often work alone. This is because most teachers are concerned about both students’ abilities to work in groups, and their abilities to work independently.

In appendix A, the model-adaptation activity that goes with the Volleyball Problem is about developing a scheme for evaluating “pocket radio recorders” for a consumer guidebook for kids.

Discussions about structural similarity are teacher-led activities that involve the whole class and that focus on structural similarities (and differences) among the constructs and conceptual systems emphasized throughout the model development sequence. A primary goal of these discussions is to provide experiences in which students go beyond thinking with these constructs and conceptual systems toward making the constructs explicit objects of thought. One way to do this is to ask questions that challenge students to identify corresponding parts of the three embodiments—and to make predictions from one embodiment to another.

Presentations and discussions are whole-class activities in which students make formal presentations about the results of work produced during either model-eliciting activities or model-adaptation activities. These results typically consist of two-page letters (or executive summaries) that provide conceptual tools that a client (who is identified in the statement of the problem) needs for a specific purpose (which also is identified in the statement of the problem). However, students also may express their results using brief multimedia presentations. In either case, the goal of the presentation and discussion session
is for students to practice explaining their work, to see examples of a variety of ways of thinking, to discuss strengths and weaknesses of alternative approaches, and to identify directions for improvement in their own work—or the work of others. In such sessions, students often get immediate feedback about the quality of their work, even without getting detailed feedback from the teacher.

As an alternative to sessions in which students make presentations about their work, teams of students also can play the role of clients who must make decisions about the strengths and weaknesses of products that various groups of students produce. More details about presentation and discussion sessions are described in chapter 9, Kay McClain in this book.

Reflection and debriefing activities often consist of brief questionnaires in which students think back about their experiences during model-eliciting activities or model-adaptation activities. Sometimes these questionnaires focus on group dynamics; and, sometimes they focus on the roles that the individual student played during different stages of solution processes. Often, the purpose of these reflection and debriefing activities is for students to express, examine, assess, and (ultimately) control their own feelings, attitudes, preferences (values), beliefs, and behaviors—without necessarily exposing themselves to their teachers or peers. Consequently, we sometimes have found it useful to program relevant questionnaires into calculators (or computers) that are able to provide instant graphic summaries that play back to students what they appear to have said—without anybody else seeing this information. In these summaries, the goal is not to label students—as though their characteristics did not vary across time, across contexts, and even across different stages during the solution of an individual problem. The goal is simply to provide reflection tools that help students assume increasingly productive personae for learning and problem solving. More details about reflection and debriefing tools are described in chapter 22 by Jim Middleton in this book.

Follow-up activities often consist of problem sets that teachers generate to help students recognize connections between their experiences during model development sequences and their experiences based on more traditional kinds of classroom activities—centered around textbooks and tests. Quite often, in modern standards-based curriculum reform projects, these follow-up activities provide a concrete way to map the goals of model development sequences to goals specified in the school districts adopted curriculum guides.

The On-Line “How To” Toolkit is a computer-based version of the kind of Schaum Outlines (workbooks) that college students use to help them survive their required courses in topic areas such as physics, chemistry algebra, calculus, and statistics (e.g., Spiegel & Stephens, 1999). That is, the “how to” tool kit is a web-based archive of “canned demonstrations”—plus problem sets and examples of solved problems that give brief explanations of facts and skills that are most frequently needed in designated topic areas. At Purdue, undergraduate students develop materials for the “how to” tool kit as part of the coursework that focuses on uses of technology in education.
Appendix D gives an example of a typical component of the “how to” tool kit that focuses on the topic of adding fractions. Other high quality resources and references include textbooks or other materials that teachers use as the basis for instruction in their classes. For example, in our own work in middle school mathematics, teachers participating in our projects often have been encouraged to use materials such as Math in Context (Romberg, 1997) or the Rational Number Project: Fraction Lessons for Middle Grades (Cramer, Post, Lesh & Behr, 1998). Sometimes teachers prefer to use these later types of materials as the foundation for their curriculum (and to use model development sequences as supplementary materials that focus on project-based learning). Or, sometimes they use project-based learning activities to provide the core of their curriculum (and use textbooks as supplementary resources). But, in either case, most teachers who have worked with us have found it useful to draw on both traditional and non-traditional resources.

COGNITIVE AND SOCIAL DIMENSIONS OF CONCEPTUAL DEVELOPMENT BOTH ARE MOST IMPORTANT IN INSTRUCTION

During model development sequences, the constructs and conceptual schemes that students develop are molded and shaped not only through interactions with concrete materials but also through interactions with other people. This is one reason why, for many of the activities within model development sequences, students work in teams with three to four students in each group.

Vygotsky’s theories (Wertsch, 1991) and social constructivist perspectives (Cobb & Yackel, 1995) often are invoked to explain the importance of student-to-student and student-to-teacher interactions in instruction. For example, in this book, chapter 9 by McClain emphasizes the importance of communication and consensus building in the development of language and other social conventions, chapter 17 by English and Lesh emphasizes similar themes related to the identification and formulation of problems, approaches, and norms for assessing solution attempts, and chapter 21 by Lesh, Lester, and Hjalmarson (2001) emphasizes external-to-internal dimensions of conceptual development. For example, this latter chapter describes how external processes of explanation and justification gradually become internalized in the sense that, when children must go beyond blind thinking to also think about thinking, they are not likely to become proficient at carrying on (internal) dialogues with themselves if they lack experience carrying on (external) dialogues with others. Similarly, they are not likely to become proficient at monitoring and assessing their own behaviors if they lack experience monitoring and assessing the behaviors of others.

For the preceding reasons, model development sequences provide experiences in which students externalize processes that otherwise would have remained internal—and internalize processes that occur first in external forms.
But, in model development sequences, justifications for small group activities derive from Piaget-influenced cognitive perspectives at least as much as from Vygotsky-influenced social perspectives. Consequently, the remaining sections of this chapter focus on cognitive justifications for social interactions—beginning the notion of “societies of mind” and Piaget’s notions of cognitive centering and cognitive egocentrism.

**CENTERING, EGOCENTRISM AND “SOCIETIES OF MIND”**

In the development of mathematical constructs and conceptual systems, the personally constructed nature of constructs is no less significant than the invented nature of social conventions—and the shared nature of knowledge. But, the construction of mathematical constructs involves not only the gradual coordination of increasingly sophisticated systems of operations and relations (which emphasize interactions with concrete materials), it also involves the coordination of perspectives (which emphasize interactions involving multiple people, or multiple perceptions by a single individual). This latter fact is true because, according to external-to-internal views of conceptual development, inter-personal interactions tend to precede intra-personal interactions—such as those that occur when a person needs to coordinate multiple ways of thinking about a single experience.

To see why the coordination of multiple perceptions becomes important in the development of mathematics constructs, it is useful to consider one of the most fundamental observations underlying modern cognitive science. That is, humans interpret learning and problem solving situations using internal constructs and conceptual systems. Consequently, their interpretations of experiences are influenced, not just by external forces and events, but also by internal models and conceptual schemes, and when learners or problem solvers attempt to match their experiences with existing conceptual schemes, some relevant information always is ignored, de-emphasized, or distorted—whereas other meanings and information are projected into the situation that are not objectively given. For example, consider the following familiar observations about eye witnesses to everyday events. In descriptions given by eye witnesses to newsworthy events (e.g., fires, traffic accidents, robberies, or questionable judgments by referees to sporting events), different witnesses often report seeing very different "facts;" and, some observations are amazingly barren, distorted, and internally inconsistent, depending on the sophistication and stability of the sense making schemes (e.g., models and conceptual systems) that they use to interpret their experiences.

Similar conceptual characteristics also apply to auditory experiences, and to other domains of experiences, as well as to visual experiences. For example, when college students listen to lectures, or when they participate in discussions with professors or peers, a great deal of what is said is likely to go unnoticed unless they have developed a stable frame of reference for interpreting what is
said, written, or shown. What one person says does not necessarily dictate what
another person hears. Students may lose cognizance of the big picture when
attention focuses on details; they may lose cognizance of details when attention
focuses on the “big picture;” or, they may lose cognizance of one type of detail
when attention focuses on other types of detail. Using terminology from
Piagetian psychology, these conceptual characteristics are referred to as: (a)
centering: noticing only the most salient information in the given situation while
ignoring other relevant information, and (b) egocentrism: distorting
interpretations to fit prior conceptions—thus attributing characteristics to the
situation that are not objectively given.

Conceptual egocentrism and centering are especially apparent when
unstable conceptual systems are used to make sense of experiences. But, to some
extent, they occur whenever any model is used to interpret another system. This
is because all models (or interpretation systems) are useful oversimplifications
of the systems they are intended to describe. They simplify (or filter out) some
aspects or reality in order to clarify (or highlight) others. Furthermore, these
facts are especially apparent when models are used to describe situations in
which:

- an overwhelming amount of information is relevant which must be
  filtered, simplified, and/or interpreted in some way in order to avoid
  exceeding human processing capabilities.
- some of the most important information has to do with patterns (trends,
  relationships, and regularities) in the available information—not just
  with isolated and unorganized bits of information.

In the preceding kinds of problem solving situations, models are used so
that meaningful patterns or relationships can be used to:

- base decisions on a minimum set of cues—because the model embodies
  an explanation of how the facts are related to one another.
- fill holes, or go beyond the filtered set of information—because the
  model gives a holistic interpretation of the entire situation, including
  hypotheses about objects or events that are not obviously given (and that
  need to be generated or sought out; Shulman, 1986).

Hallmarks of students early (and often unstable) conceptual systems include
the facts that: (a) within a given interpretation, it often is difficult to keep forest-
level and tree-level perspectives in mind at the same time, and (b) interpretations
tend to shift unconsciously from one tree-level perspective to another, or from
one forest-level perspective to a corresponding tree-level perspective—and then
back to another forest-level perspective. Consequently, regardless of whether the
problem solver is an individual or a group, a community of loosely related
conceptual schemes (or fragments of conceptual schemes) tends to be available
to interpret nearly any given situation; and, the result is a “society of mind”
(Minsky, 1987) in which meaningful communication is needed among
participants in a team—or among competing interpretations within a given individual.

Later in this book, chapter 19 by Zawojewski, Lesh, and English gives more details about cognitive perspectives concerning why (and what kind of) group interactions may be productive in model development sequences. In the meantime, the main points to emphasize here are that: (a) the preceding "society of mind" characteristic of real life modeling processes leads to some of the most interesting social dimensions of understanding that are relevant to model development sequences, and (b) one of the main instructional design principles that we use to address these social dimensions of understanding is called the multiple perspective principle.

THE MULTIPLE PERSPECTIVE PRINCIPLE

According to the Dienes’ original version of the multiple embodiment principle, a student should investigate a series of structurally similar embodiments (as shown in Fig. 2.2), and the goal is to go beyond thinking with the relevant organizational/relational/operational system to also think about the system. But, when relevant conceptual systems are not yet functioning as well coordinated systems-as-a-whole, students tend to focus on surface-level characteristics of the concrete materials rather than on underlying patterns and regularities of the conceptual systems applied to these materials, and their thinking tends to be characterized by centering and conceptual egocentrism; that is, (a) the meanings that students associate with their experiences often are remarkably barren and distorted, (b) the student often has difficulty noticing more than a small number of surface characteristics of relevant experiences, and (c) the things students see in a given activity often varies a great deal from one moment to another—as attention shifts from one perspective to another, from the big picture to details, or from one type of detail to another.

Students with the preceding conceptual characteristics generally have difficulty recognizing structure-related similarities and differences among potentially related embodiments, and they also have difficulties coordinating related perspectives of a single problem-solving experience. Consequently, to help such students, the multiple perspective principle suggests that it often is useful to put students in situations where several alternative perspectives (or “windows”) are juxtaposed for a single problem solving experience (see Fig. 2.5a), and/or each student interacts with other students who are centering and distorting in ways that differ from their own (see Fig. 2.5b).
By closely juxtaposing multiple perspectives of a single situation, a goal is to encourage students to notice more and to distort less within any given perspective—and to think more deeply about experiences. For example:

- Teams of students may work together who have different technical capabilities, different cognitive styles, or different prior experiences.
- Within group problem solving episodes, students may be encouraged to play different roles—such as manager, monitor, recorder, data gatherer, or tool operator.
- After problem-solving episodes are completed, students may use reflection tools to think about group functioning, or about roles played by different individuals, or about ideas and strategies that were and were not productive.
- After problem-solving episodes have been completed, students may serve on editorial boards that assess strengths and weaknesses of results that other groups produce, or they may play the role of the client who needs the result that the problem solvers were asked to produce.

Other techniques for juxtaposing multiple perspectives are discussed in greater detail in chapter 19 of this book, on a models and modeling perspective about social dimensions of conceptual development (Zawojewski, Lesh, & English, chap. 19, this volume). In the meantime, a point to emphasize is that simply juxtaposing multiple perspectives does not guarantee that these perspectives interact—nor that conflicting perspectives are recognized, nor that mismatches lead to cognitive conflicts of the type that Piaget believed to be important mechanisms to propel conceptual growth (Piaget & Beth, 1966). In the same way that a gradual process is required for students to coordinate their ways of thinking into coherent conceptual systems-as-a-whole, gradual processes also are needed to coordinate multiple perspectives of a single experience. Nonetheless, it often is possible to facilitate these processes by externalizing processes that otherwise would
have been internal—by noticing, for example, that students may be able to monitor the behaviors of others before they can monitor their own behaviors. In general, this approach is based on Vygotsky's notion of a *zone of proximal development* which focuses on the gradual internalization of external processes (Wertsch, 1985).

An ultimate goal of the multiple perspective principle is to help individual students behave, within themselves, as though they were three people working together around a table—so that it is easier for them to overcome the barren and distorted inadequacies that are inherent in their own early interpretations of learning and problem solving situations.

**SUMMARY**

The kind of model-development sequences described in this chapter are designed to be used in research, as well as in assessment or instruction. Furthermore, they are designed to focus on deeper and higher-order understandings of the conceptual schemes that underlie a small number of especially powerful constructs in elementary mathematics—rather than on trying to cover a large number of small facts and skills. Yet, skill-level abilities are not neglected because it is relatively easy to link model development sequences to instructional materials that deal with these latter types of ideas and abilities. In particular, model development sequences are modularized to make it easy for researchers or teachers to add, delete, modify, or re-sequence their components.

For example:

- It is possible to use model-eliciting activities (i.e., model-construction activities) as stand-alone problem solving experiences—perhaps being preceded by a warm-up activity and followed by student presentations or discussions focusing on response assessment.

- It is possible to use model-eliciting activities (or model-adaptation activities) as performances assessments—and to use them somewhat like pre-tests (or post-tests) preceding (or following) a traditional instructional unit (or a chapter in a book) in which the relevant construct is emphasized. In this case, warm-up and follow-up activities might not be used.
• It is possible to have students engage in a complete model-development sequence—and to use traditional paper-based or computer-based materials as supplementary resources, where needed, for specific students to focus on specific facts and skills.

In general, in the preceding sequences, model-eliciting activities and model-exploration activities are designed for students to work together in teams consisting of three to four students each; other activities, such as the presentations and discussions, also are intended to emphasize student-to-student or student-to-teacher interactions as much as interactions with concrete materials. In fact, even in model-adaptation activities, where students often work individually, rather than in teams, the goal is to develop conceptual tools that are sharable, transportable, and reusable for a variety of purposes in a variety of situations. Therefore, important social dimensions of conceptual development are not neglected.

Unlike constructivist teaching materials in which carefully guided sequences of questions provide the only means of leading students to assemble and adopt conceptual systems of the type the author has in mind, model development sequences put students in situations where they must express, test, and modify, revise, and refine their own ways of thinking during the process of designing powerful conceptual tools that embody constructs that students are intended to develop. In short, students adapt their own ways of thinking rather than adopting the author’s (or teacher’s) ways of thinking, and the adaptation (modification, extension, and revision) of existing conceptual systems is given as much attention as the construction (or assembly) of conceptual systems that are assumed to be completely new to the student(s).
Sometimes, it is useful for students to invent their own language, diagrams, metaphors, or notation systems that express their ways of thinking—in the presentations and discussions that follow model-eliciting activities, as well as in other activities that occur in model development sequences. But, in other cases, and especially in model-exploration activities (where a primary goal is to introduce students to powerful language, diagrams, metaphors, or notation systems), it often is not necessary to expect students to invent conventions that took many years to develop in the history of mathematics and science.

Concerning the artificial introduction of socially accepted language, symbols, and other representational media, dangers that arise result from the facts that it is easy to introduce language and symbols whose meanings presuppose the existence of conceptual schemes that students have not yet developed. The result is that students often sound like they understand constructs that they in fact do not. But, this reason for being conservative about the introduction of standard language, notations, and conventions is a very different than the notion that the value of these media depends mainly on votes from some group of people—rather than depending on their power and utility that they provide for the underlying conceptual systems they are intended to embody.

In addition to focusing on powerful constructs and conceptual systems, the kind of activities that are emphasized in Model development sequences are intended to go beyond isolated problem-solving experiences that are intended mainly as vehicles for emphasizing problem-solving processes.